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NEW YORK UNIVERSITY
Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

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Structure of the Boundary Layer in a Maxwellian Plasma

MARTIN H. MILLMAN and JAMES HURLEY

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AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
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Abstract

The problem considered here is that of the transition region separating a uniform Maxwellian Plasma from its confining magnetic field. The geometry chosen is one-dimensional. The magnetic field profile within the boundary layer is determined by the current distribution, i.e., by the paths of the ions and electrons. The paths of these particles on the other hand are determined by the fields in which they move. The equation of motion of the ions is treated exactly and the electrons in the guiding center approximation. A self consistent solution of Maxwell's equations is obtained. These results are compared with those obtained for a mono-energetic distribution and found to agree to within 10%.

1. Introduction

This report is an extension of an earlier work of one of the authors[†]. The problem discussed there is that of the transition region separating a uniform plasma and its confining magnetic field. It was possible to perform the calculations exactly due to the simplifying restriction that the plasma be mono-energetic, i.e., that all particles have the same energy. If both the electron and ion component are treated exactly the analysis is quite laborious. On the other hand if the ions are treated exactly and the electrons in the guiding center approximation the error involved in the transition layer thickness is only 5 parts in 10^6.

In the present paper we should like to consider a more realistic plasma--a plasma which becomes Maxwellian far from the transition region. Unfortunately it is not possible to solve this problem exactly. However the agreement between the exact and approximate calculation for the mono-energetic plasma suggests that the approximate calculations should suffice for the Maxwellian plasma as well.

We shall not dwell here on the formulation of the problem or upon the nature of the approximation as these points are discussed in detail in I. We shall content ourselves with a brief statement of the problem and a presentation of the results.

2. Statement of the Problem

All quantities are assumed to depend only on a single variable x.

[†] J. Hurley, Physics of Fluids 6, 83 (1963). Hereafter referred to as I.

The magnetic field is directed everywhere in the z direction and is uniform from $x = -\infty$ to $x = 0$. As we cross the plane $x = 0$ we enter into the domain of the collision free plasma where the magnetic field decreases and eventually becomes uniform again at $x = +\infty$ where the plasma is required to be Maxwellian. We further require that the ion density be identically equal to the electron density throughout the plasma domain. (Thus there are no electric fields.) The paths of the particles are determined by the magnetic field in which they move. The field in turn is determined by the paths of the particles. The problem is to solve, self-consistently, the equations of motion of the particles (Liouville's equation) and Maxwell's equations.

3. Analysis

The magnetic field is determined by the relation

$$\frac{dB}{dx} = \frac{-4\pi}{c}(j_+ + j_-) \tag{1}$$

where j_+ and j_- are the ion and electron currents and are given by

$$j_\pm = \pm e \int \dot{y}\, f^*_\pm (x, \dot{x}, \dot{y}, \dot{z})\, d\dot{x}\, d\dot{y}\, d\dot{z} \tag{2}$$

and f^*_+ and f^*_- are the ion and electron distribution functions. The distribution functions will satisfy Liouville's equation if they are chosen to be functions of the constants of the motion.

These constants are

$$E_{\pm} = \frac{1}{2} m_{\pm} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$P_{\pm} = m_{\pm} \dot{y} \pm e A/c \qquad (3)$$

$$Q_{\pm} = m_{\pm} \dot{z}$$

where $A(x)$ is the vector potential defined by the relation $B = dA/dx$ and normalized so that $A(0) = 0$. We write $f(E,P,Q)$ for $f^*(x,\dot{x},\dot{y},\dot{z})$ when expressed as a function of E, P, and Q. We shall choose as the ion distribution function

$$f_{+} = (m_{+}/2\pi kT)^{3/2} \; N_{+}(\infty) \; \eta \, (P^2 - 2m_{+}E) \; e^{-E/kT} \qquad (4)$$

where η is the unit step function and $N_{+}(\infty)$ is the ion density at $x = +\infty$ (as may be seen by integrating f_{+}). The variables in (4) are restricted by the inequality

$$2 m_{+}E - (P - \frac{eA}{c})^2 - Q^2 \geq 0$$

It may be seen from (2) that this is equivalent to

$$\dot{x}^2 \geq 0$$

Such a choice for the ion distribution function requires (1) that there shall be no particles in the domain x 0 (or equivalently in the domain

$A < 0$ since we assume that $A(x)$ is a monotonic function of x) and (2) that the distribution shall approach a Maxwellian distribution as x approaches infinity.

We may now compute the ion current required to solve the differential equation (1). Substituting (4) into (2) we find

$$j_+ = (e/m)\, N_+(\infty) \sqrt{2\, m_+ kT}\; \alpha\, (\,1 - \text{Prob}\;\alpha) \tag{5}$$

where

$$\alpha = eA/c\; \sqrt{8 m_+ kT} \tag{6}$$

and

$$\text{Prob}\;\alpha = \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-u^2}\, du$$

However to compute the electron current it is necessary to find f_-, the electron distribution function. This may be done by solving the integral equation obtained by equating the ion and electron densities

$$N_{\pm} = \int f^*_{\pm}\, (x, \dot{x}, \dot{y}, \dot{z})\; d\dot{x}\; d\dot{y}\; d\dot{z} \tag{7}$$

Unfortunately this is too difficult to solve when f_+ is given by (4). It was shown in I that it is possible to bypass this step when the guiding center approximation is used for the electrons, for then

$$j_- = e \frac{dp_{xx}^{(-)}}{d\,A} = e\,k\,T \frac{d\,N_-}{d\,A} = e\,k\,T \frac{d\,N_+}{d\,A} \qquad (8)$$

where $p_{xx}^{(-)}$ is the indicated component of the electron stress tensor. We use (4) and (7) to determine N_+ and substitute into (8) giving

$$j_- = \frac{ekT\,N_+\,(\infty)}{\sqrt{2\pi m_+ kT}} \; e^{-\alpha^2} \qquad (9)$$

Introducing (5) and (9) into (1) we obtain the ordinary differential equation for the magnetic field

$$\frac{d^2\,\alpha}{d\,\xi^2} = -\left[\alpha\,(\,1 - \text{Prob}\,\alpha\,) + e^{-\alpha^2} / \,2\,\sqrt{\pi}\,\right] \qquad (10)$$

where

$$\xi = \sqrt{2\,\pi\,e^2\,N_+\,(\infty)\,/\,m_+ c^2}\quad x \qquad (11)$$

The initial conditions $A(0) = 0$ and $B(0) = B_o$ become

$$\alpha\,(0) = 0 \quad , \left(\frac{d\,\alpha}{d\,\xi}\right)_{\xi=0} = B_o \,/\, \sqrt{16\,\pi\,N_+\,(\infty)\,kT} \qquad (12)$$

We shall find it more convenient to parametrize the plasma by its field at $x = \infty$ (B_∞) rather than by $N_+\,(\infty)\,kT$. This may be accomplished by integrating (10) once from $\xi = 0$ to $\xi = \infty$. After some simplifications we obtain

$$\frac{B_o^2}{8\pi} = \frac{B_\infty^2}{8\pi} + 2\, N_+\,(\infty)\, kT \tag{13}$$

which we recognize as the condition of equilibrium for a large parallelopiped whose left face lies in the vacuum at $x = 0$ where the pressure is purely magnetic and equal to $B_o^2/8\pi$ and whose right face lies in the Maxwellian plasma at $x = \infty$ where the field pressure is $B_\infty^2 / 8\pi$ and the ion pressure $N_+(\infty)kT$ and the electron pressure $N_-(\infty)\; kT = N_+(\infty)\; kT$. Observing that $B_o = (dA/dx)_{x=0}$ we may rewrite the second initial condition (12) in the form

$$\left[\frac{d\,\alpha}{d\,\xi}\right]_{\xi=0} = \sqrt{\frac{B_o^2}{B_o^2 - B_\infty^2}}$$

For purposes of comparing these results for the Maxwellian plasma with those obtained in I for the mono-energetic plasma we have integrated (10) - (12) numerically for the special case $B_\infty = B_c/\sqrt{2}$. The results are shown in figures (1) and (2). As a measure of the transition layer thickness we chose that point at which the field has fallen by .9 of the total discontinuity. This occurs at $\xi = 1.15$ or

$$x_o = 1.15\; \sqrt{m_+ c^2/2\pi e^2 N_+\;(\infty)} = 4.60\; c\; \sqrt{m_+ kT}\; /e\; B_o$$

where we have made use of (11) and (13). Now the average energy per particle is $E_o = 3/2\; kT$. In terms of this energy we may write for the transition layer thickness

$$x_o = 3.77\ c\ \sqrt{m_+ E_o}\ /\ e\ B_o$$

This is to be compared with the exact solution for the mono-energetic plasma

$$x_o = 3.30\ c\ \sqrt{m_+ E_+}\ /\ e\ B_o$$

where E_+ is the kinetic energy common to all particles. We conclude then that the depth of the transition layer is essentially insensitive to the energy distribution in the plasma.

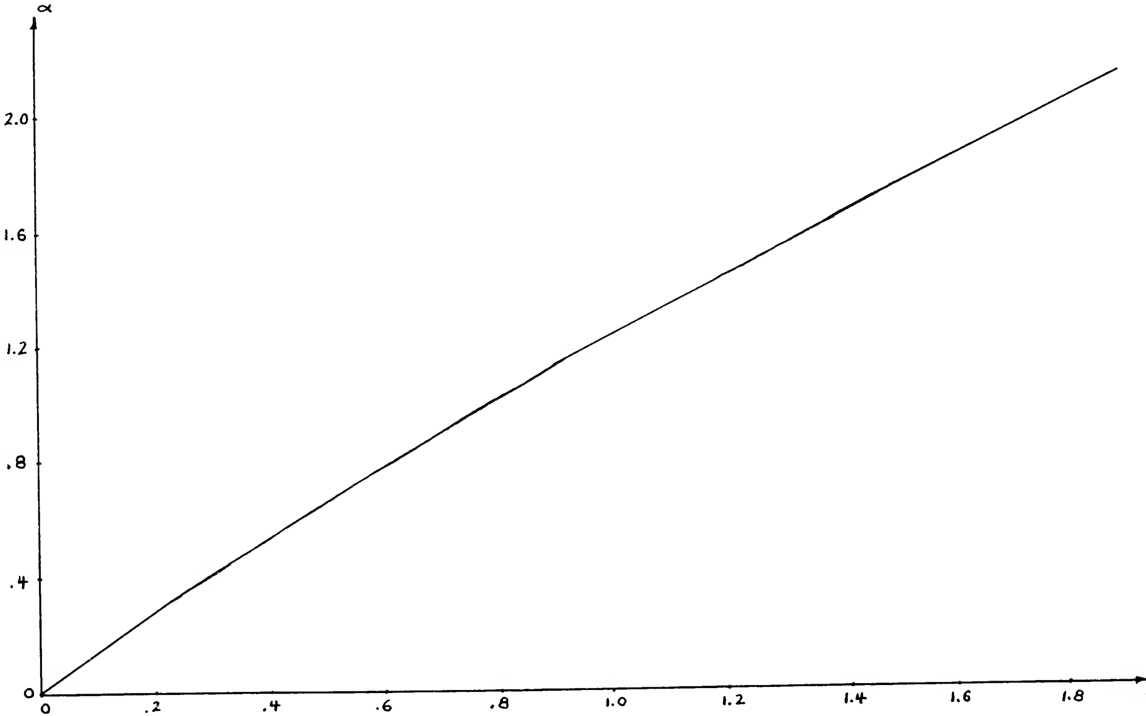

Fig. (1)

Profile of the Vector Potential

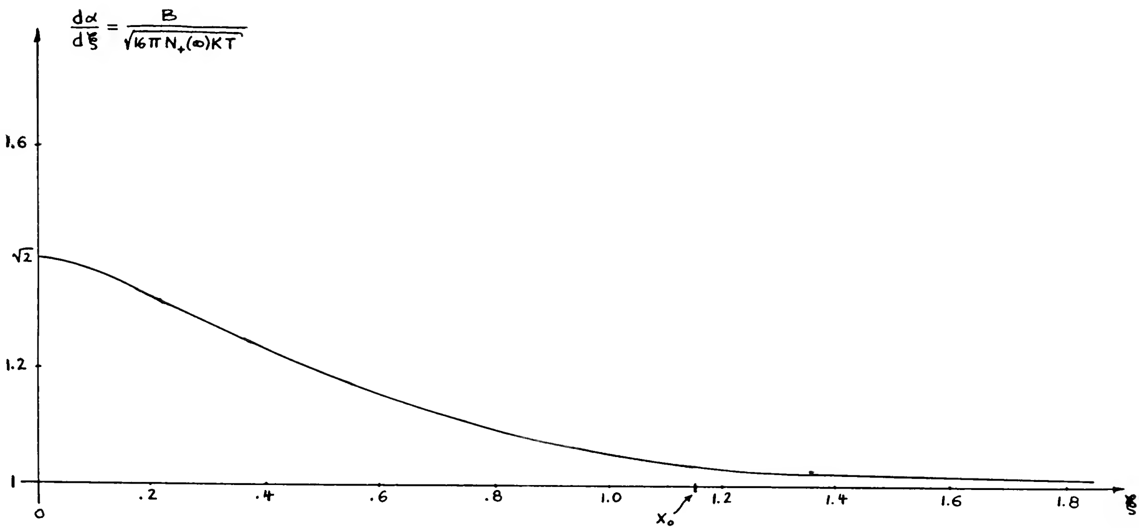

Fig. (2)

Profile of the Magnetic Field

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2. Magnetohydrodynamics

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